Steady Flow of an Electrically Conducting Incompressible Viscoelastic Fluid over a Heated Plate

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The transformation group theoretic approach is applied to the problem of the flow of an electrically conducting incompressible viscoelastic fluid near the forward stagnation point of a heated plate. The application of one-parameter transformation group reduces the number of independent variables, by one, and consequently the basic equations governing flow and heat transfer are reduced to a set of ordinary differential equations. These equations have been solved approximately subject to the relevant boundary conditions by employing the shooting numerical technique. The effect of the magnetic parameter M, the Prandtl number Pr and the non-dimensional elastic parameter representing the non-Newtonian character of the fluid k on velocity field, shear stress, temperature distribution and heat flux are carefully examined.

Key words: One-parameter Transformation Group; Viscoelastic Fluid; Non-Newtonian Fluid.

1. Introduction

Many attempts have been made to study the flow properties of non-Newtonian fluids. In 1969 Soundalgekar and Puri [1] have considered the interesting version of the problem of fluctuating flow of a non-Newtonian viscoelastic fluid under the condition of very small elastic parameter. The equations of motion for the steady state yield a third order non-linear differential equation, when the elasticity effect is taken into consideration, to be solved subject to two boundary conditions only. To overcome this difficulty, they [1] used a method which was developed by Beard and Walters [2] in 1964. They obtained the approximate solution valid for sufficiently small values of the elastic parameter by employing a perturbation procedure. The heat transfer aspect of this problem has been investigated by Massoudi and Ramezan [3] in 1992 and Garg [4] in 1994. In 1990 Garg and Rajagopal [5] and in 1994 Garg [4] have obtained solutions valid for all values of an elastic parameter by using an additional boundary condition at infinity, whereas Massoudi and Ramezan's [3] work is confined to small values of elastic parameter.

In the present work we consider the flow of an electrically conducting incompressible viscoelastic fluid near the forward stagnation point of a solid plate. This type of problems has applications to engineering processes and polymer technology. The main purpose of this work is to study the effect of the magnetic parameter, the Prandtl number and the non-dimensional elastic parameter representing the non-Newtonian character of the fluid on velocity field, shear stress, temperature distribution and heat flux.

Similarity solutions are convenient methods to reduce systems of partial differential equations into systems of manageable ordinary differential equations. The mathematical technique used in the present analysis which leads to a similarity representation of the problem is the one-parameter group transformation. Group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkhoff [6] in 1948, who made use of one-parameter transformation groups. Moran and Gaggioli [7, 8], in 1966 and 1969, presented a theory which has led to improvements over earlier similarity methods. Similarity analysis has been applied intensively by Gabbert [9] in 1967. For more additional discussions on group transformation, see Ames [10, 11], Bluman and Cole [12], Boisvert et al. [13], Gaggioli and Moran [14, 15]. Throughout the history of similarity analysis, a variety of problems in science and engineering has been solved. Many physical applications are illustrated by Abd-el-Malek et al. [16, 17].

2. Mathematical Formulation of the Problem

In terms of the stream function ψ the boundary layer equations for a steady flow of an electrically conducting incompressible viscoelastic near the forward stagnation point of a solid plate are given by [18] in the form

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{\mathrm{d}U}{\mathrm{d}x}+\frac{\partial^3\psi}{\partial y^3}-M\frac{\partial\psi}{\partial y} -k\Big[\frac{\partial\psi}{\partial y}\frac{\partial^4\psi}{\partial x\partial y^3}-\frac{\partial\psi}{\partial x}\frac{\partial^4\psi}{\partial y^4}+\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}\Big], \tag{2.1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{1}{Pr}\frac{\partial^2 T}{\partial y^2}, \tag{2.2}$$

where M is the magnetic parameter, k is a small non-dimensional elastic parameter representing the non-Newtonian character of the fluid, Pr is the Prandtl number and $U(x)$ denotes the streamwise velocity component. The boundary conditions are

$$y=0:\quad \frac{\partial\psi}{\partial y}=0,\quad \frac{\partial\psi}{\partial x}=0,\quad T=T_0, \tag{2.3}$$

$$y\to\infty:\quad \lim_{y\to\infty}\frac{\partial\psi}{\partial y}=U(x),\quad \lim_{y\to\infty}T=0, \tag{2.4}$$

where T_0 is constant.

It is noticed that (2.1), characterizing the flow, has one derivative with respect to x and four derivatives with respect to y, but there are only three boundary conditions. To obtain a solution, we need two extra boundary conditions. To overcome this requirement of additional conditions, we seek a solution of (2.1) using the perturbation analysis [19] in the form

$$\psi=\psi_0+k\psi_1+O(k^2), \tag{2.5}$$

where ψ_0 and ψ_1 are the first and second approximation of the stream function.

Therefore (2.1) – (2.4) can be converted to

$$\frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_0}{\partial x\partial y}-\frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_0}{\partial y^2} =U\frac{\mathrm{d}U}{\mathrm{d}x}+\frac{\partial^3\psi_0}{\partial y^3}-M\frac{\partial\psi_0}{\partial y}, \tag{2.6}$$

$$\frac{\partial\psi_1}{\partial y}\frac{\partial^2\psi_0}{\partial x\partial y}+\frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_1}{\partial x\partial y}-\frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_1}{\partial y^2} -\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}=\frac{\partial^3\psi_1}{\partial y^3}-M\frac{\partial\psi_1}{\partial y} -\Big[\frac{\partial\psi_0}{\partial y}\frac{\partial^4\psi_0}{\partial x\partial y^3}-\frac{\partial\psi_0}{\partial x}\frac{\partial^4\psi_0}{\partial y^4}+\frac{\partial^2\psi_0}{\partial x\partial y}\frac{\partial^3\psi_0}{\partial y^3}-\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial^3\psi_0}{\partial x\partial y^2}\Big], \tag{2.7}$$

$$\frac{\partial\psi_0}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial T}{\partial y}+k\Big[\frac{\partial\psi_1}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial T}{\partial y}\Big] =\frac{1}{Pr}\frac{\partial^2 T}{\partial y^2}, \tag{2.8}$$

subject to

$$y=0:\quad \frac{\partial\psi_0}{\partial y}=0,\quad \frac{\partial\psi_0}{\partial x}=0,\quad \frac{\partial\psi_1}{\partial y}=0,\quad \frac{\partial\psi_1}{\partial x}=0,\quad T=T_0, \tag{2.9}$$

$$y\to\infty:\quad \lim_{y\to\infty}\frac{\partial\psi_0}{\partial y}=U(x),\quad \lim_{y\to\infty}\frac{\partial\psi_1}{\partial y}=0,\quad \lim_{y\to\infty}T=0. \tag{2.10}$$

Notice that, as $k\to 0:\ \psi\to\psi_0$ and hence ψ_0 must satisfy the same conditions as ψ.

3. Solution of the Problem

The method of solution depends on the application of a one-parametric group transformation to the system of partial differential equations (2.6) – (2.8). Under this transformation, the two independent variables will be reduced by one and the differential equations (2.6) – (2.8) transforms into a system of ordinary differential equations in only one independent variable, which is the similarity variable.

3.1. The Group Systematic Formulation

The procedure is initiated with the group G, a class of transformation of one parameter a of the form

$$G:\bar{S}=C^s(\mathrm{a})S+P^s(\mathrm{a}), \tag{3.1}$$

where S stands for x, y, ψ_0, ψ_1, U, T and the C's and P's are real-valued functions and at least differentiable in the real argument a.

3.2. The Invariance Analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations:

$$\overline{S_{\bar{i}}} = \left(\frac{C^s}{C^i}\right) S_i, \quad \overline{S_{\bar{i}\bar{j}}} = \left(\frac{C^s}{C^i C^j}\right) S_{ij}, \qquad i = x, y;\; j = x, y. \tag{3.2}$$

where S stands for ψ_0, ψ_1, U and T and $S_i = \frac{\partial S}{\partial i}$, $S_{ij} = \frac{\partial^2 S}{\partial i \partial j}, \ldots$.

Equations (2.6) – (2.8) are said to be invariantly transformed whenever

$$\begin{aligned}
&\frac{\partial \bar{\psi}_0}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}_0}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}_0}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}_0}{\partial \bar{y}^2} - \bar{U} \frac{\mathrm{d}\bar{U}}{\mathrm{d}\bar{x}} - \frac{\partial^3 \bar{\psi}_0}{\partial \bar{y}^3} + M \frac{\partial \bar{\psi}_0}{\partial \bar{y}} \\
&= H_1(\mathrm{a}) \Big[\frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} \\
&\qquad - \frac{\mathrm{d}U}{\mathrm{d}x} - \frac{\partial^3 \psi_0}{\partial y^3} + M \frac{\partial \psi_0}{\partial y} \Big],
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\frac{\partial \bar{\psi}_1}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}_0}{\partial \bar{x} \partial \bar{y}} + \frac{\partial \bar{\psi}_0}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}_1}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}_0}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}_1}{\partial \bar{y}^2} - \frac{\partial \bar{\psi}_1}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}_0}{\partial \bar{y}^2} - \frac{\partial^3 \psi_1}{\partial \bar{y}^3} \\
&+ M \frac{\partial \bar{\psi}_1}{\partial \bar{y}} - \Big(\frac{\partial \bar{\psi}_0}{\partial \bar{y}} \frac{\partial^4 \bar{\psi}_0}{\partial \bar{x} \partial \bar{y}^3} - \frac{\partial \bar{\psi}_0}{\partial \bar{x}} \frac{\partial^4 \psi_0}{\partial \bar{y}^4} + \frac{\partial^2 \bar{\psi}_0}{\partial \bar{x} \partial \bar{y}} \frac{\partial^3 \bar{\psi}_0}{\partial \bar{y}^3} - \frac{\partial^2 \bar{\psi}_0}{\partial \bar{y}^2} \frac{\partial^3 \bar{\psi}_0}{\partial \bar{x} \partial \bar{y}^2} \Big) \\
&= H_2(\mathrm{a}) \Big[\frac{\partial \psi_1}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} + \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_1}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial^3 \psi_1}{\partial \bar{y}^3} \\
&\qquad + M \frac{\partial \psi_1}{\partial y} - \Big(\frac{\partial \psi_0}{\partial y} \frac{\partial^4 \psi_0}{\partial x \partial y^3} - \frac{\partial \psi_0}{\partial x} \frac{\partial^4 \psi_0}{\partial y^4} + \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial^3 \psi_0}{\partial y^3} - \frac{\partial^2 \psi_0}{\partial y^2} \frac{\partial^3 \psi_0}{\partial x \partial y^2} \Big) \Big],
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\frac{\partial \bar{\psi}_0}{\partial \bar{y}} \frac{\partial \bar{T}}{\partial \bar{x}} - \frac{\partial \bar{\psi}_0}{\partial \bar{x}} \frac{\partial \bar{T}}{\partial \bar{y}} + k \Big(\frac{\partial \bar{\psi}_1}{\partial \bar{y}} \frac{\partial \bar{T}}{\partial \bar{x}} - \frac{\partial \bar{\psi}_1}{\partial \bar{x}} \frac{\partial \bar{T}}{\partial \bar{y}} \Big) - \frac{1}{Pr} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \\
&\qquad = H_3(\mathrm{a}) \Big[\frac{\partial \psi_0}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi_0}{\partial x} \frac{\partial T}{\partial y} + k \Big(\frac{\partial \psi_1}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi_1}{\partial x} \frac{\partial T}{\partial y} \Big) - \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2} \Big],
\end{aligned} \tag{3.5}$$

for some functions $H_1(\mathrm{a})$, $H_2(\mathrm{a})$ and $H_3(\mathrm{a})$ which depend only on the group parameter a.

Substitution from (3.1) into (3.3) – (3.5) for the independent variables, the functions and their derivatives yields

$$\begin{aligned}
&\frac{(C^{\psi_0})^2}{C^x (C^y)^2} \Big(\frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} \Big) - \frac{(C^U)^2}{C^x} U \frac{\mathrm{d}U}{\mathrm{d}x} - \frac{C^{\psi_0}}{(C^y)^3} \frac{\partial^3 \psi_0}{\partial y^3} + \frac{C^{\psi_0}}{C^y} M \frac{\partial \psi_0}{\partial y} + R_1(\mathrm{a}) \\
&\qquad = H_1(\mathrm{a}) \left[\frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial^3 \psi_0}{\partial y^3} + \frac{M}{x} \frac{\partial \psi_0}{\partial y} \right],
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\frac{C^{\psi_0} C^{\psi_1}}{C^x (C^y)^2} \Big(\frac{\partial \psi_1}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} + \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_1}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} \Big) \\
&- \frac{(C^{\psi_0})^2}{C^x (C^y)^4} \Big(\frac{\partial \psi_0}{\partial y} \frac{\partial^4 \psi_0}{\partial x \partial y^3} - \frac{\partial \psi_0}{\partial x} \frac{\partial^4 \psi_0}{\partial y^4} + \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial^3 \psi_0}{\partial y^3} - \frac{\partial^2 \psi_0}{\partial y^2} \frac{\partial^3 \psi_0}{\partial x \partial y^2} \Big) - \frac{C^{\psi_1}}{(C^y)^3} \frac{\partial^3 \psi_1}{\partial y^3} + \frac{C^{\psi_1}}{C^y} M \frac{\partial \psi_1}{\partial y} + R_2(\mathrm{a}) \\
&\qquad = H_2(\mathrm{a}) \Big[\frac{\partial \psi_1}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} + \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_1}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial^3 \psi_1}{\partial y^3} \\
&\qquad\qquad + M \frac{\partial \psi_1}{\partial y} - \Big(\frac{\partial \psi_0}{\partial y} \frac{\partial^4 \psi_0}{\partial x \partial y^3} - \frac{\partial \psi_0}{\partial x} \frac{\partial^4 \psi_0}{\partial y^4} + \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial^3 \psi_0}{\partial y^3} - \frac{\partial^2 \psi_0}{\partial y^2} \frac{\partial^3 \psi_0}{\partial x \partial y^2} \Big) \Big],
\end{aligned} \tag{3.7}$$

and

$$\frac{C^{\psi_0}C^T}{C^xC^y}\Big(\frac{\partial\psi_0}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial T}{\partial y}\Big)+k\frac{C^{\psi_1}C^T}{C^xC^y}\Big(\frac{\partial\psi_1}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial T}{\partial y}\Big)-\frac{1}{Pr}\frac{C^T}{(C^y)^2}\frac{\partial^2 T}{\partial y^2}+R_3(\mathrm{a})$$
$$=H_3(\mathrm{a})\Big[\frac{\partial\psi_0}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial T}{\partial y}+k\Big(\frac{\partial\psi_1}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial T}{\partial y}\Big)-\frac{1}{Pr}\frac{\partial^2 T}{\partial y^2}\Big], \tag{3.8}$$

where

$$R_1(\mathrm{a})=P^U\frac{C^U}{C^x}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{3.9}$$

$$R_2(\mathrm{a})=0, \tag{3.10}$$

and

$$R_3(\mathrm{a})=0. \tag{3.11}$$

The invariance of (3.6)–(3.8) implies $R_1(\mathrm{a})=R_2(\mathrm{a})=R_3(\mathrm{a})\equiv 0$. This is satisfied by putting

$$P^U=0, \tag{3.12}$$

and

$$\frac{(C^{\psi_0})^2}{C^x(C^y)^2}=\frac{(C^U)^2}{C^x}=\frac{C^{\psi_0}}{(C^y)^3}=\frac{C^{\psi_0}}{C^y}=H_1(\mathrm{a}), \tag{3.13}$$

$$\frac{(C^{\psi_0})C^{\psi_1}}{C^x(C^y)^2}=\frac{(C^{\psi_0})^2}{C^x(C^y)^4}=\frac{C^{\psi_1}}{(C^y)^3}=\frac{C^{\psi_1}}{C^y}=H_2(\mathrm{a}), \tag{3.14}$$

$$\frac{C^{\psi_0}C^T}{C^xC^y}=\frac{C^{\psi_1}C^T}{C^xC^y}=\frac{C^T}{(C^y)^2}=H_3(\mathrm{a}), \tag{3.15}$$

which yields

$$C^y=1,\quad C^{\psi_0}=C^x,\quad C^{\psi_1}=C^x,\quad \text{and}\quad C^U=C^x. \tag{3.16}$$

Moreover, the boundary conditions (2.9) and (2.10) are also invariant in form, that implies

$$P^y=P^T=0\quad\text{and}\quad C^T=1. \tag{3.17}$$

Finally, we get the one-parameter group G which transforms invariantly, the differential equations (3.6)–(3.8) and the boundary conditions (2.9) and (2.10). The group G is of the form

$$G:\begin{cases}\bar{x}=C^x x+P^x,\\ \bar{y}=y,\\ \bar{\psi}_0=C^x\psi_0+P^{\psi_0},\\ \bar{\psi}_1=C^x\psi_1+P^{\psi_1},\\ \bar{U}=C^xU,\\ \bar{T}=T.\end{cases} \tag{3.18}$$

3.3. The Complete Set of Absolute Invariants

Our aim is to make use of group methods to present the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in analysis to obtain a complete set of absolute invariants. In addition to the absolute invariant of the independent variable, there are four absolute invariants of the dependent variables ψ_0, ψ_1, U and T.

If $\eta=\eta(x,y)$ is the absolute invariant of the independent variables, then

$$g_j(x,y;\psi_0,\psi_1,U,T)=F_j[\eta(x,y)],\quad j=1,2,3,4, \tag{3.19}$$

which are the four absolute invariants corresponding to ψ_0, ψ_1, U and T. A function $g=g(x,y;\psi_0,\psi_1,U,T)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation

$$\sum_{i=1}^{6}(\alpha_iS_i+\beta_i)\frac{\partial g}{\partial S_i}=0, \tag{3.20}$$

where S_i stands for x, y; ψ_0, ψ_1, U and T, and

$$\alpha_i=\frac{\partial C^{S_i}}{\partial \mathrm{a}}(\mathrm{a}^0),\quad \beta_i=\frac{\partial P^{S_i}}{\partial \mathrm{a}}(\mathrm{a}^0),\quad i=1,2,\ldots,6, \tag{3.21}$$

where a^0 denotes the value of a which yields the identity element of the group.

From group (3.18) and using (3.21), we get:

$$\alpha_2=\beta_2=\beta_5=\alpha_6=\beta_6=0.$$

At first, we seek the absolute invariant of the independent variables. Owing to (3.20), $\eta(x,y)$ is an absolute invariant if it satisfies the first-order linear partial differential equation

$$(\alpha_1x+\beta_1)\frac{\partial\eta}{\partial x}+(\alpha_2y+\beta_2)\frac{\partial\eta}{\partial y}=0,$$

which reduces to

$$\frac{\partial \eta}{\partial x} = 0. \tag{3.22}$$

Equation (3.22) has a solution of the form

$$\eta(x,y) = y. \tag{3.23}$$

Similarly, analysis of the absolute invariants of the dependent variables ψ_0, ψ_1, U and T are

$$\left.\begin{aligned} \psi_0(x,y) &= \Gamma_0(x)F_0(\eta), \\ \psi_1(x,y) &= \Gamma_1(x)F_1(\eta), \\ U(x) &= \Gamma_2(x), \\ T(x,y) &= \theta(\eta). \end{aligned}\right\} \tag{3.24}$$

3.4. The Reduction to Ordinary Differential Equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain an ordinary differential equation. Generally, the absolute invariant η has the form given in (3.23).

Substituting from (3.24) into (2.6) yields

$$\begin{aligned} &\frac{\mathrm{d}^3F_0}{\mathrm{d}\eta^3} + \frac{\mathrm{d}\Gamma_0}{\mathrm{d}x}\left[F_0\frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2} - \left(\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right)^2\right] \\ &-M\frac{\mathrm{d}F_0}{\mathrm{d}\eta} + \frac{\Gamma_2}{\Gamma_0}\frac{\mathrm{d}\Gamma_2}{\mathrm{d}x} = 0. \end{aligned} \tag{3.25}$$

For (3.25) to be reduced to an expression in a single independent variable η, the coefficients in (3.25) should be constants or functions of η. Thus,

$$\frac{\mathrm{d}\Gamma_0}{\mathrm{d}x} = C_1, \tag{3.26}$$

$$\frac{\Gamma_2}{\Gamma_0}\frac{\mathrm{d}\Gamma_2}{\mathrm{d}x} = C_2. \tag{3.27}$$

Assume $C_1 = 1$ and $C_2 = U_0$, where U_0 is an arbitrary constant, then $\Gamma_0 = x$, and therefore $\Gamma_2(x) = U_0 x$ that actually obeys the power-law fluids. Hence, (3.25) reduces to

$$\frac{\mathrm{d}^3F_0}{\mathrm{d}\eta^3} + F_0\frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2} - \left(\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\right)^2 - M\frac{\mathrm{d}F_0}{\mathrm{d}\eta} = -U_0^2. \tag{3.28}$$

Substitute from the above results and from (3.24) into (2.7), we obtain

$$\begin{aligned} &\frac{\mathrm{d}^3F_1}{\mathrm{d}\eta^3} + F_0\frac{\mathrm{d}^2F_1}{\mathrm{d}\eta^2} - (1+M)\frac{\mathrm{d}F_1}{\mathrm{d}\eta} \\ &+\frac{x}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}x}\left(\frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2}F_1 - \frac{\mathrm{d}F_1}{\mathrm{d}\eta}\right) \\ &= \frac{x}{\Gamma_1}\left[-F_0\frac{\mathrm{d}^4F_0}{\mathrm{d}\eta^4} + \frac{\mathrm{d}F_0}{\mathrm{d}\eta}\frac{\mathrm{d}^3F_0}{\mathrm{d}\eta^3} - \left(\frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2}\right)^2\right]. \end{aligned} \tag{3.29}$$

Again, for (3.29) to be reduced to an expression in a single independent variable η, the coefficients in (3.29) should be constants or functions of η. Thus,

$$\frac{x}{\Gamma_1} = C_3, \tag{3.30}$$

$$\frac{x}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}x} = C_4. \tag{3.31}$$

Assuming $C_3 = 1$, then $\Gamma_1 = x$, and therefore $C_4 = 1$, from which (3.29) takes the form

$$\begin{aligned} &\frac{\mathrm{d}^3F_1}{\mathrm{d}\eta^3} + F_0\frac{\mathrm{d}^2F_1}{\mathrm{d}\eta^2} - \left(2\frac{\mathrm{d}F_0}{\mathrm{d}\eta} + M\right)\frac{\mathrm{d}F_1}{\mathrm{d}\eta} + \frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2}F_1 \\ &= -F_0\frac{\mathrm{d}^4F_0}{\mathrm{d}\eta^4} + 2\frac{\mathrm{d}F_0}{\mathrm{d}\eta}\frac{\mathrm{d}^3F_0}{\mathrm{d}\eta^3} - \left(\frac{\mathrm{d}^2F_0}{\mathrm{d}\eta^2}\right)^2. \end{aligned} \tag{3.32}$$

Finally, using (3.24), (2.8) will be converted to the following ordinary differential equation

$$\frac{d^2\theta}{\mathrm{d}\eta^2} + Pr\,F\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = 0, \tag{3.33}$$

where $F = F_0 + kF_1$. Thus, under the similarity variable η, (2.6)–(2.8) and their boundary conditions (2.9) and (2.10) will be transformed into the system of differential equations (3.28), (3.32) and (3.33) with the following appropriate corresponding conditions

$$\begin{aligned} &\eta = 0: \quad F_0(\eta) = 0, \quad \frac{\mathrm{d}F_0(\eta)}{\mathrm{d}\eta} = 0, \\ &F_1(\eta) = 0, \quad \frac{\mathrm{d}F_1(\eta)}{\partial\eta} = 0, \quad \theta(\eta) = T_0, \end{aligned} \tag{3.34}$$

$$\begin{aligned} &\eta \to \infty: \quad \lim_{y\to\infty}\frac{\mathrm{d}F_0(\eta)}{\mathrm{d}\eta} = U_0, \\ &\lim_{y\to\infty}\frac{\mathrm{d}F_1(\eta)}{\mathrm{d}\eta} = 0, \quad \lim_{y\to\infty}\theta(\eta) = 0. \end{aligned} \tag{3.35}$$

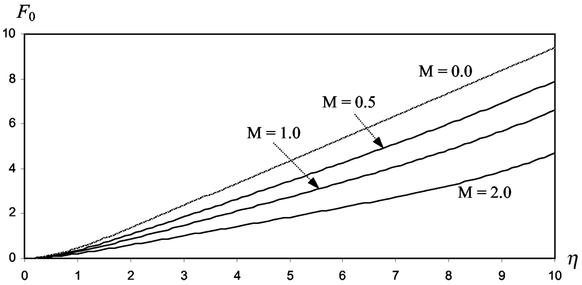

Fig. 1a. Effect of M on the first approximation of stream function profiles at $Pr = 0.7$ and $k = 0.2$.

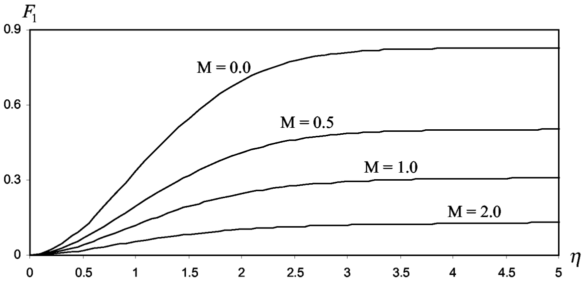

Fig. 2a. Effect of M on the second approximation of stream function profiles at $Pr = 0.7$ and $k = 0.2$.

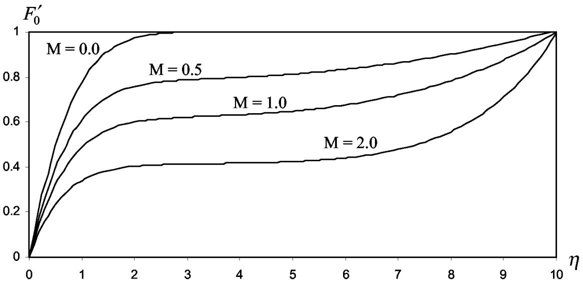

Fig. 1b. Effect of M on the first approximation of velocity profiles at $Pr = 0.7$ and $k = 0.2$.

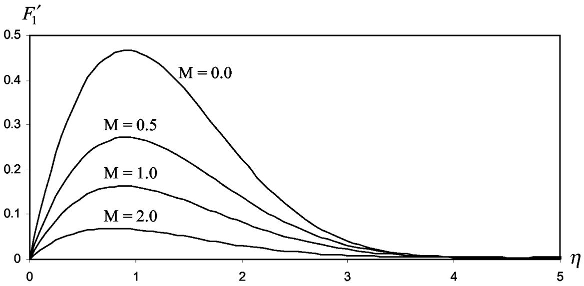

Fig. 2b. Effect of M on the second approximation of velocity profiles at $Pr = 0.7$ and $k = 0.2$.

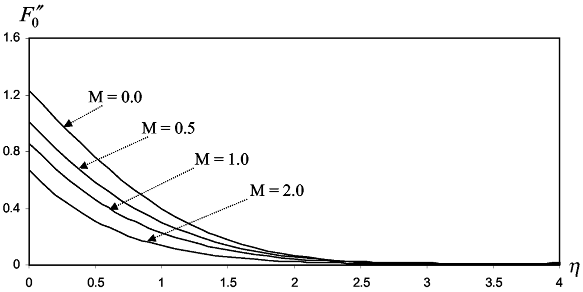

Fig. 1c. Effect of M on the first approximation of shear stress profiles at $Pr = 0.7$ and $k = 0.2$.

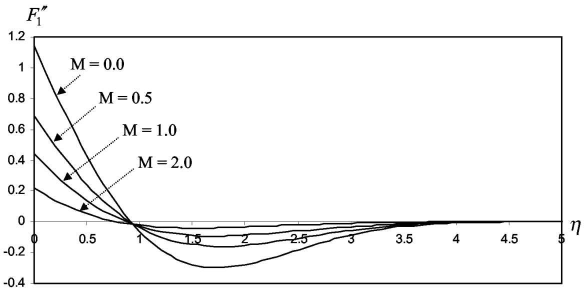

Fig. 2c. Effect of M on the second approximation of shear stress profiles at $Pr = 0.7$ and $k = 0.2$.

4. Numerical Results

For convenience let $U_0 = T_0 = 1$, the set of boundary value problem represented by (3.28), (3.32) and (3.33) under the appropriate conditions (3.34) and (3.35) has been solved numerically using the fourth-order Runge-Kutta shooting method. Having found $F = F_0 + kF_1$ from (3.28) and (3.32), the solution for (3.33) subject to its relevant conditions is obtained by a similar shooting method. We have an initial value problem from $\eta_0 = 0$ to η_∞, where η_∞ is a sufficiently large number.

Figure 1a shows the first approximation of stream function F_0 as a function of the similarity variable η, for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is noticed that the stream function F_0 decreases and comes close to each other as the magnetic parameter M increases.

Figure 1b shows the variation of first approximation of the velocity F_0' with η for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is clear that the velocity of the fluid decreases with increasing the magnetic parameter. In addition, this figure shows that the smaller the value of M the faster it reaches the maximum value of F_0'.

Figure 1c shows the variation of first approximation of the shear stress F_0'' with η for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is obvious that the shear stress changes depending on the magnetic parameter and the distance; the shear stress decreases with increasing the magnetic parame-

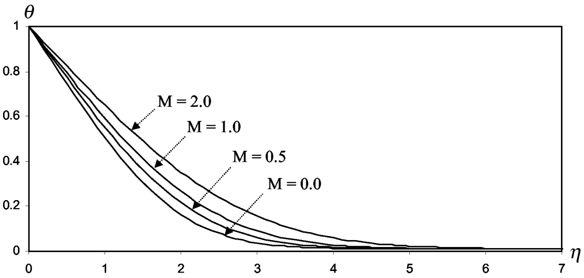

Fig. 3a. Effect of magnetic parameter M on temperature profiles at $Pr = 0.7$ and $k = 0.2$.

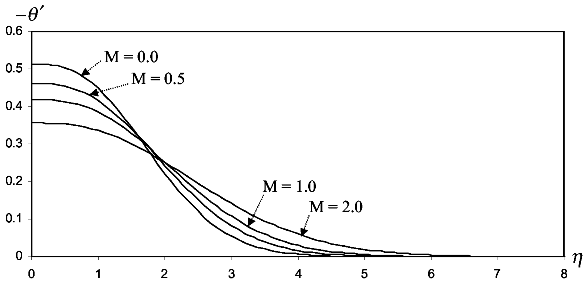

Fig. 3b. Effect of magnetic parameter M on heat flux profiles at $Pr = 0.7$ and $k = 0.2$.

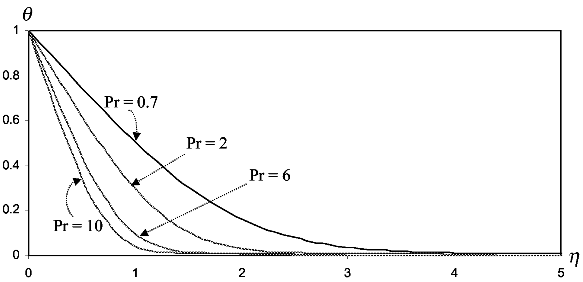

Fig. 4a. Effect of Pr on temperature profiles at $M = 0$ and $k = 0.2$.

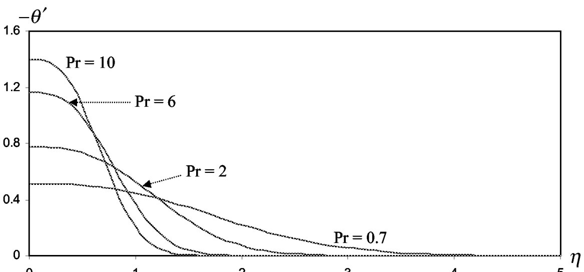

Fig. 4b. Effect of Pr on heat flux profiles at $M = 0$ and $k = 0.2$.

ter. For a small value of M, the shear stress starts with a high value then decreases with increasing the distance. For a high value of M, the shear stress starts with a lower value and decreases with the distance.

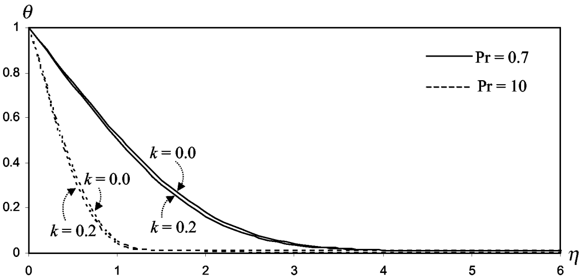

Fig. 5a. Effect of elastic parameter thickness on temperature profiles at $M = 0$.

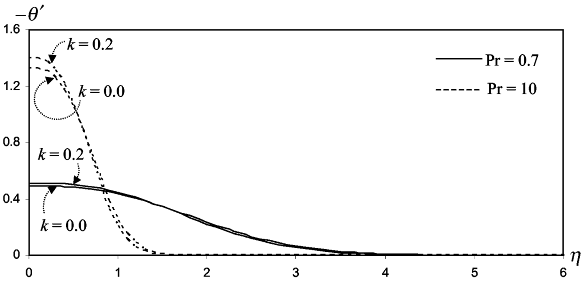

Fig. 5b. Effect of elastic parameter thickness on heat flux profiles at $M = 0$.

Figure 2a shows the second approximation of stream function F_1 as a function of the similarity variable η, for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is noticed that the stream function F_1 decreases as the magnetic parameter M increases. It seems that F_1 approaches zero as M tends to infinity.

Figure 2b shows the variation of second approximation of the velocity F_1' with η for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is clear that F_1' overshoots for small values of η and then decreases to zero for large η. In addition, this figure shows that the smaller the value of M the faster it reaches the maximum value of F_1'.

Figure 2c shows the variation of second approximation of the shear stress F_1'' with η for various values of magnetic parameter M and for fixed $Pr = 0.7$ and $k = 0.2$. It is obvious that F_1'' decreases with increasing the magnetic parameter for small values of η and then becomes negative in a certain region and increases to zero for large η.

The variation of the temperature θ with η is illustrated in Figure 3a. The results are obtained for $M = 0$, 0.5, 1, 2 and corresponding to $Pr = 0.7$ and $k = 0.2$. The figure shows the rapid decrease of the temperature distribution at $M = 0$. Also, the temperature increases with increasing the magnetic parameter M.

Figure 3b shows the variation of the heat flux $-\theta'$ with η. It is clear that the heat flux starts with a higher value for the lower values of the magnetic parameter M and then decreases.

The effect of Pr on the temperature and heat flux is illustrated in Figs. 4a and 4b. The results are obtained for $Pr = 0.7$, 2, 6 and 10. For the temperature profile, Fig. 4a indicates the occurrence of the rapid decrease in θ. This becomes more evident for larger values of Pr. Also, Fig. 4b shows the rapid decrease in the heat flux for increasing values of Pr.

Figure 5a shows the variation of the temperature θ with η and Fig. 5b shows the variation of the heat flux $-\theta'$ with η for various values of non-dimensional elastic parameter k. It is apparent from Fig. 5a that the temperature profiles slightly decrease with an increase in the elasticity of the fluid, but in Fig. 5b the heat flux distribution varies (higher-lower) at different values of the elasticity of the fluid. Again from Figs. 5a and 5b, we arrive to the conclusion that the effect of non-dimensional elastic parameter k is still small for the increase in the Prandtl number.

5. Conclusion

The group method confirmed that it is a powerful tool for solving the problems of magneto-elastic-viscous flow near the forward stagnation point of a solid plate with heat transfer and obtaining the velocity profiles, shear stress and heat flux for various values of the magnetic parameter. Numerical results of the transformed boundary layer equations have been obtained by using the Runge-Kutta shooting method. Referring to the numerical results and the figures it is observed that:

(i) We observe from Figs. 1 and 2 that the main effect of increasing the magnetic parameter M on the two dimensional magneto-elastic-viscous flow is to decrease the velocity field and shear stress in the direction of the solid plate.

(ii) From Fig. 5, we arrive at the conclusion that the thermal boundary layer thickness becomes small for the increase in the Prandtl number Pr.

(iii) Our perturbation analysis is valid only for small values of elastic parameter k.

Acknowledgement

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